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Some Results on Exhaustiveness in Asymmetric Metric Spaces

Zeynep Hande Toyganözü*[a], Serpil Pehlivan[a]

[a] *Süleyman Demirel University, Faculty of Arts and Sciences, Department of Mathematics, Isparta, Turkey*

Abstract. We introduce here the notion of exhaustiveness, which is related with the notion of equicontinuity, in asymmetric metric spaces. We give the relation between equicontinuity and exhaustiveness in such spaces and some theorems and results about it. We show that in the asymmetric situation forward convergence does not imply backward convergence (or vice versa), the limit of a sequence of exhaustive functions may not be continuous, also may not be unique. Also, we prove a type of Ascoli theorem using the notion of exhaustiveness in the asymmetric case. Finally, following Caserta and Kočinac [3], we will investigate some properties of a statistical version of exhaustiveness in asymmetric metric spaces.

1. Introduction and preliminaries

Asymmetric metric spaces were first introduced by Wilson [26] in 1931 as quasi-metric spaces, and then studied by many authors (see, for instance, [1, 16, 18, 23]). An asymmetric metric space is a generalization of a metric space but the symmetry axiom is eliminated in the definition of metric spaces. We can come up with some troubles in several classic statements of symmetric analysis without the symmetry property in the definition of such spaces. In asymmetric metric spaces some notions, such as convergence, completeness and compactness are different from the metric case. There are two notions for each of them, namely forward and backward ones, since we have two topologies which are the forward topology and the backward topology in the same space (see [15]). Collins and Zimmer [6] studied these notions in the asymmetric context.

An example that asymmetric metrics are common in real life is taxicab geometry topology including one-way streets, where can have a path from point A to point B contains a different set of streets than a path from B to A. Also, there can be found the latest applications of asymmetric metric spaces in the field of pure and applied mathematics and material science as in [4, 17–19]. In [5], Cobzaş gave the basic results on asymmetric normed spaces.

Gregoriades and Papanastassiou [13] introduced the notion of exhaustiveness for families and sequences of functions which is close to the notion of equicontinuity.

In this study, we first introduce the notions of forward and backward exhaustiveness in asymmetric metric spaces. We investigate the properties of exhaustiveness and its relation with equicontinuity in such spaces. Also, by giving some examples, we show the importance of forward and backward convergences in the asymmetric situation for some statements about the limit of a sequence of exhaustive functions. Then,

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* Corresponding author
Email addresses: handetoyganozu@sdu.edu.tr (Zeynep Hande Toyganözü*), serpilpehlivan@sdu.edu.tr (Serpil Pehlivan)

we will give a characterization of compactness of $\mathcal{F}$ using exhaustiveness in asymmetric metric spaces. In the last section, following Caserta and Kočinac ([3]), we will investigate some properties of a statistical version of exhaustiveness in asymmetric metric spaces.

1.1. Asymmetric metric spaces

Let us recall some definitions and results on asymmetric metric spaces which were given in [6].

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is an *asymmetric metric* and (X, d) is an *asymmetric metric space* if:
(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$, for every $x, y \in X$,
(2) $d(x, z) \leq d(x, y) + d(y, z)$, for every $x, y, z \in X$.

Definition 1.2. The *forward topology* τ_+ induced by d is the topology generated by the forward open balls $B^+(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for $x \in X, \varepsilon > 0$.

Likewise, the *backward topology* τ_- induced by d is the topology generated by the backward open balls $B^-(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$ for $x \in X, \varepsilon > 0$.

Definition 1.3. A set $S \subset X$ is *forward bounded* (resp. *backward bounded*), if there exists $x \in X$ and $\varepsilon > 0$ such that $S \subset B^+(x, \varepsilon)$ (resp. $S \subset B^-(x, \varepsilon)$).

Definition 1.4. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *forward convergent* to $x \in X$ (*backward convergent* to $x \in X$) if and only if

$$\lim_{n\to\infty} d(x, x_n) = 0 \ (\lim_{n\to\infty} d(x_n, x) = 0)$$

and is denoted by $x_n \xrightarrow{f} x$ ($x_n \xrightarrow{b} x$).

Definition 1.5. A sequence $(x_n)_{n\in\mathbb{N}}$ in an asymmetric metric space (X, d) is *forward Cauchy* (*backward Cauchy)* if for each $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for $k \geq n \geq N$, $d(x_n, x_k) < \varepsilon$ ($d(x_k, x_n) < \varepsilon$) holds.

Definition 1.6. Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. A function $f : X \to Y$ is said to be ff*–continuous* (fb*–continuous*) at $x \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x, \delta)$ implies $f(y) \in B^+(f(x), \varepsilon)$, ($f(y) \in B^-(f(x), \varepsilon)$).

Definition 1.7. (Sequential definition of continuity) A function $f : X \to Y$ is ff*–continuous* at $x \in X$ if and only if whenever $x_k \xrightarrow{f} x$ in (X, d_X) we have $f(x_k) \xrightarrow{f} f(x)$ in (Y, d_Y).

The statement holds analogously for the other types.

In the following proposition, Collins and Zimmer [6] gave the relation of forward and backward limits.

Proposition 1.8. *Let $d(y, x) \leq k(x, y)d(x, y)$ for every $x, y \in X$, where $k : X \times X \to \mathbb{R}$ satisfies the following constraint:*

$$\forall x \in X \ \exists \varepsilon > 0 \textit{ such that } y \in B^+(x, \varepsilon) \Rightarrow k(x, y) \leq K(x). \tag{1}$$

Therefore, the existence of forward limits implies the existence of backward limits, and so limits are unique.

Example 1.9. The Sorgenfrey asymmetric metric is the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ is given by

$$d(x, y) := \begin{cases} y - x, \text{if } y \geq x \\ 1, \text{if } y < x \end{cases}$$

which does not satisfy (1).

Definition 1.10. A set $S \subset X$ is

(1) *forward compact* if every open cover of S in the forward topology has a finite subcover.

(2) *forward relatively compact* if $\overline{S}$ is forward compact, where $\overline{S}$ denotes the closure of S in the forward topology.

(3) *forward sequentially compact* if every sequence in X contains a forward convergent subsequence.

(4) *forward complete* if every forward Cauchy sequence is forward convergent.

Lemma 1.11. *Let $d : X \times X \to \mathbb{R}_0^+$ be an asymmetric metric. If (X, d) is forward sequentially compact and $x_k \xrightarrow{b} x$, then $x_k \xrightarrow{f} x$.*

Lemma 1.12. *A forward compact set X is forward sequentially compact.*

Definition 1.13. A subset $S \subset X$ is *forward totally bounded* if for each $\varepsilon > 0$ it can be covered by finitely many forward balls of radius ε.

Proposition 1.14. *If (X, d) is forward sequentially compact and forward totally bounded, then X is forward compact.*

Definition 1.15. Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. A set $\mathcal{F}$ of functions from X to Y is *forward equicontinuous* (*backward equicontinuous*) if for every $x \in X$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in Y$ and every $f \in \mathcal{F}$ with $d_X(x, y) < \delta$, $d_Y(f(x), f(y)) < \varepsilon$ ($d_Y(f(y), f(x)) < \varepsilon$) holds.

Let Y^X be the set of functions from X to Y and $C(X, Y)$ be the set of all $ff-$continuous functions from X to Y. The uniform metric on Y^X is

$$\overline{\rho}(f, g) := \sup\{\overline{d}(f(x), g(x)) \Big| x \in X\},$$

where $\overline{d}(x, y) := \min\{d(x, y), 1\}$ and d is the asymmetric metric associated with Y. This metric induces the uniform topology on Y^X.

Definition 1.16. Let $(f_n)_{n\in\mathbb{N}}$ be a function sequence and f be a function from X to Y.

(i) We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *forward pointwise convergent* (*backward pointwise convergent*) to f if for every $\varepsilon > 0$ there exist a $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d_Y(f(x), f_n(x)) < \varepsilon$ ($d(f_n(x), f(x)) < \varepsilon$).

(ii) We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *forward convergent uniformly* (*backward convergent uniformly)* with limit f if for every $\varepsilon > 0$ there exists a natural number N such that for all $x \in X$ and all $n \geq N$ we have $d(f(x), f_n(x)) < \varepsilon$ ($d(f_n(x), f(x)) < \varepsilon$).

1.2. Statistical convergence

The concept of statistical convergence of a sequence was introduced at a conference held at Wroclaw University, Poland, in 1949, by Steinhaus [25] (see also [11]). Since then it has been studied by many authors (see, for instance [2, 7, 9, 12, 20, 22, 24]). This idea is based on the notion of asymptotic density of a set $A \subset \mathbb{N}$. In [8], Das and Bhunia studied the idea of statistical convergence of double sequences for real numbers in asymmetric metric spaces.

Let $A \subset \mathbb{N}$. The number of elements which are less than or equal to some $n \in \mathbb{N}$ and which belong to A is denoted by $|A(n)|$. The *asymptotic density* of A is defined by $\partial(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$, provided that this limit exists. Note that $\partial(\mathbb{N} \setminus A) = 1 - \partial(A)$ if $\partial(A)$ exists. A set is said to be statistically dense if $\partial(A) = 1$.

Let X be an asymmetric metric space. A real sequence $(x_n) \in X$ is said to be *forward statistically convergent* to $x \in X$, (*backward statistically convergent* to $x \in X$) provided that for every $\varepsilon > 0$

$$\partial(\{n \in \mathbb{N} : d(x, x_n) \geq \varepsilon\}) = 0 \ (\partial(\{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon\}) = 0\ )$$

and is denoted by $x_n \xrightarrow{f-st} x$ ($x_n \xrightarrow{b-st} x$).

2. Forward and backward exhaustiveness

In [13], Gregoriades and Papanastassiou gave the notion of exhaustiveness for family and sequences of functions. Similarly, in this section we give the notion of exhaustiveness in asymmetric metric spaces and investigate some of its properties.

Definition 2.1. Let $(X, d_X), (Y, d_Y)$ be asymmetric metric spaces, $x \in X$, let $\mathcal{F}$ be a family of functions from X to Y and $f_n : X \to Y, n \in \mathbb{N}$.

(i) If $\mathcal{F}$ is infinite, then we say that the family $\mathcal{F}$ is *forward exhaustive* (*backward exhaustive*) at a point $x \in X$ provided that, for each $\varepsilon > 0$ there exists a $\delta > 0$ and a finite subset K of $\mathcal{F}$ such that $d_Y(f(x), f(y)) < \varepsilon$ ($d_Y(f(y), f(x)) < \varepsilon$) for all $f \in \mathcal{F} \setminus K$ and all $y \in X$ such that $y \in B^+(x, \delta)$.

(ii) If $\mathcal{F}$ is finite, then we say that the family $\mathcal{F}$ is forward exhaustive at $x \in X$ provided that each member of $\mathcal{F}$ is ff−continuous at $x \in X$.

(iii) The sequence $(f_n)_{n\in\mathbb{N}}$ is called *forward exhaustive* (*backward exhaustive*) at $x \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $y \in B^+(x, \delta)$ and for all $n \geq n_0$ we have $d_Y(f_n(x), f_n(y)) < \varepsilon$ ($d_Y(f_n(y), f_n(x)) < \varepsilon$). The sequence $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive (backward exhaustive) if it is forward exhaustive (backward exhaustive) at every $x \in X$.

It is clear that forward equicontinuity implies forward exhaustiveness. If for every $\varepsilon > 0$ the finite set K in Definition 2.1.(i) can be taken to be the empty set, then a forward equicontinuous family is forward exhaustive.

But the converse is not true. Forward exhaustiveness of $\mathcal{F}$ does not imply that there exists a finite subset K of $\mathcal{F}$ such that $\mathcal{F} \setminus K$ is forward equicontinuous, since the set K in the definition depends on $\varepsilon > 0$.

Similar remarks mentioned above can be given for backward ones.

Proposition 2.2. *Let $(X, d_X), (Y, d_Y)$ be asymmetric metric spaces, $x \in X$, $\mathcal{F}$ be an infinite family of functions from X to Y. Then, we have the following:*

(i) $\mathcal{F}$ is forward equicontinuous at x if and only if $\mathcal{F}$ is forward exhaustive at x and f is ff−continuous at x for each $f \in \mathcal{F}$.

(ii) $\mathcal{F}$ is backward equicontinuous at x if and only if $\mathcal{F}$ is backward exhaustive at x and f is fb−continuous at x for each $f \in \mathcal{F}$.

Proof. It is enough to prove (i) only, since the proof of (ii) will be similar. The necessity part is clear from the definitions.

For the converse part, let $\varepsilon > 0$, then there exist $\delta' > 0$ and K finite subset of $\mathcal{F}$ such that for every $y \in B^+(x, \delta')$ and for every $f \in \mathcal{F} \setminus K$ we have $d_Y(f(x), f(y)) < \varepsilon$. Since each f is ff− continuous at x, there exists $\delta(f) > 0$ such that for every $y \in B^+(x, \delta(f))$ we have $d_Y(f(x), f(y)) < \varepsilon$. Put $\delta := \min\{\delta', \delta(f) : f \in K\} > 0$. Then for every $y \in B^+(x, \delta)$ and for every $f \in \mathcal{F}$ we have $d_Y(f(x), f(y)) < \varepsilon$, which completes the proof. □

The following example shows that each member of a forward exhaustive family (or sequence) need not be continuous.

Example 2.3. Let $X = \mathbb{R}$ and $Y = [0, 1/2]$ be given with the following asymmetric metrics:

$$d_X(x, y) = d_Y(x, y) = \begin{cases} y - x, & \text{if } y \geq x \\ 1, & \text{if } y < x. \end{cases}$$

Let $f_n : X \to Y, n \in \mathbb{N}$, defined by

$$f_n(x) = \begin{cases} \dfrac{1}{3n}, & \text{if } x \leq 0 \\[2ex] \dfrac{1}{2n}, & \text{if } x > 0. \end{cases}$$

Notice that none of f_n is continuous at $x = 0$. But, the sequence $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive at $x = 0$. Indeed, let $\varepsilon > 0$, so there exists $n_0 \in \mathbb{N}$, $n_0 > \dfrac{1}{6\varepsilon}$ such that for $\delta > 0$, for all $y \in B^+(0,\delta) = [0,\delta)$ and for all $n \geq n_0$ we have $d_Y(f_n(0), f_n(y)) = f_n(y) - f_n(0) = \dfrac{1}{2n} - \dfrac{1}{3n} = \dfrac{1}{6n} < \varepsilon$.

So we obtained as a result of this example that there exists a forward exhaustive family which has no continuous functions.

The following lemma gives the relationship between forward exhaustiveness and backward exhaustiveness. Since this lemma is a generalization of a statement for equicontinuity by Collins and Zimmer [6] to exhaustiveness, the proof can be given in a similar way.

Lemma 2.4. *Let (Y, d_Y) be a forward compact asymmetric metric space and forward convergence implies backward convergence in Y. If a set $\mathcal{F} \subset Y^X$ is forward exhaustive, then it is also backward exhaustive (and vice versa).*

If the condition of implication of forward convergence to backward convergence is dropped in this lemma, forward exhaustiveness does not imply backward exhaustiveness.

Example 2.5. Let us consider the previous example. Forward convergence does not imply backward convergence in the asymmetric metric on Y and the sequence $(f_n)_{n\in\mathbb{N}}$ is not backward exhaustive at $x = 0$. Indeed, for $\delta > 0$ and for all $y \in B^+(0,\delta) = [0,\delta)$ we have $d_Y(f_n(y), f_n(0)) = 1$.

Now, let the asymmetric metric on Y be as in the following:

$$d_Y(x,y) = \begin{cases} y - x, & \text{if } y \geq x \\ \frac{1}{2}(x - y), & \text{if } y < x. \end{cases}$$

In this asymmetric metric, forward convergence implies backward convergence.

Let $\varepsilon > 0$. There exists a $\delta > 0$ and $n_0 \in \mathbb{N}, n_0 > \dfrac{1}{6\varepsilon}$ such that for all $y \in B^+(0,\delta) = [0,\delta)$ and for all $n \geq n_0$ we have

$$d_Y(f_n(0), f_n(y)) = f_n(y) - f_n(0) = \frac{1}{2n} - \frac{1}{3n} = \frac{1}{6n} < \varepsilon.$$

So, the sequence $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive at $x = 0$.

Now, we will show that the sequence $(f_n)_{n\in\mathbb{N}}$ is backward exhaustive at $x = 0$. Let $\varepsilon > 0$. There exist a $\delta > 0$ and $n_0 \in \mathbb{N}$, $n_0 > \dfrac{1}{12\varepsilon}$ such that for all $y \in B^+(0,\delta) = [0,\delta)$ and for all $n \geq n_0$ we have

$$d_Y(f_n(y), f_n(0)) = \frac{1}{2}(f_n(y) - f_n(0)) = \frac{1}{2}\left(\frac{1}{2n} - \frac{1}{3n}\right) = \frac{1}{2}\frac{1}{6n} < \varepsilon.$$

So, the sequence $(f_n)_{n\in\mathbb{N}}$ is backward exhaustive at $x = 0$.

By the following theorem, it can be stated that the forward pointwise limit of a sequence of forward exhaustive function is continuous, only if forward convergence implies backward convergence in the asymmetric metric on Y.

Theorem 2.6. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces such that forward convergence implies backward convergence in Y and let $f, f_n, n \in \mathbb{N}$, be functions from X to Y. If $(f_n)_{n\in\mathbb{N}}$ is forward pointwise convergent to f and $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive at $x \in X$, then f is $ff-$continuous at $x \in X$.*

Proof. Fix $\varepsilon > 0$. Since, $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive at $x \in X$ there exist a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $y \in B^+(x,\delta)$ and for all $n \geq n_0$ we have $d_Y(f_n(x), f_n(y)) < \varepsilon/3$. Let $y \in B^+(x,\delta)$. Since $(f_n)_{n\in\mathbb{N}}$ is forward pointwise convergent to f, there exists an $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $d_Y(f(y), f_n(y)) < \varepsilon/3$ and

$d_Y(f(x), f_n(x)) < \varepsilon/3$. Because forward convergence implies backward convergence in Y, $d_Y(f_n(y), f(y)) < \varepsilon/3$ holds. Now, put $N := \max\{n_0, n_1\}$. So , if $y \in B^+(x, \delta)$,

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) \\ &< \varepsilon. \end{aligned}$$

Therefore, f is $ff-$continuous at $x \in X$. □

The next example shows that if forward convergence does not imply backward convergence in Y, then the forward pointwise limit of a sequence of forward exhaustive function may not be continuous, and also may not be unique.

Example 2.7. Let $X := [0, 1]$ be equipped with the Euclidean metric and let $Y := \{(y_1, y_2) : y_1 = 0\ y_2 \in (0, 1]\} \cup \{(-1, 0)\} \cup \{(1, 0)\} \subset \mathbb{R}^2$. Choose the metric on Y to be the Sorgenfrey metric in Example 1.9, extended such that for (y_1, y_2) with $y_2 > 0$,

$$d((y_1, y_2), (\pm 1, 0)) = d((\pm 1, 0), (\mp 1, 0)) = 1,\ d((\pm 1, 0), (y_1, y_2)) = y_2$$

Let $f : X \to Y, x \to \begin{cases} (-1, 0), x = 0 \\ (1, 0), x > 0 \end{cases}$ and $f_n : X \to Y, x \to (0, \dfrac{1}{n})$.

The sequence $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive and pointwise converges forwardly to discontinuous limit f. However, the limit is not unique, since forward convergence does not imply backward convergence. For example, $(f_n)_{n\in\mathbb{N}}$ is forward pointwise convergent to $\widetilde{f} : X \to Y, x \to (1, 0)$ being continuous.

A similar argument yields the following result.

Corollary 2.8. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces such that backward convergence implies forward convergence in Y and $f, f_n, n \in \mathbb{N}$, be functions from X to Y. If $(f_n)_{n\in\mathbb{N}}$ is backward pointwise convergence to f and $(f_n)_{n\in\mathbb{N}}$ is backward exhaustive at $x \in X$, then f is $fb-$continuous at $x \in X$.*

Theorem 2.9. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces such that forward convergence implies backward convergence in Y and $f, f_n, n \in \mathbb{N}$, be functions from X to Y. If the function sequence $(f_n)_{n\in\mathbb{N}}$ is forward pointwise convergence to f and $(f_n)_{n\in\mathbb{N}}$ is forward exhaustive at X, then the function sequence $(f_n)_{n\in\mathbb{N}}$ is forward convergent uniformly to f on every forward compact subset of X.*

Proof. Let $K \subset X$ be any forward compact set and choose $\varepsilon > 0$ arbitrarily and let $x \in K$. Since,the function sequence $(f_n)_{n\in\mathbb{N}}$, is forward exhaustive at x, then there exist a $\delta_x > 0$ and $n_0 \in \mathbb{N}$ such that for all $y \in B^+(x, \delta_x)$ and for all $n \geq n_0$ we have $d_Y(f_n(x), f_n(y)) < \varepsilon/3$.

By Theorem 2.6, f is $ff-$continuous at x, therefore there exists a $\delta > 0$ with $\delta_x < \delta$ such that $d_Y(f(x), f(y)) < \varepsilon/3$ holds for all $y \in B^+(x, \delta_x)$. Since forward convergence implies backward convergence in Y, we have $d_Y(f(y), f(x)) < \varepsilon/3$.

Since $X \subseteq \bigcup\limits_{x\in X} B^+(x, \delta_x)$ and X is forward compact, there exist some $x_1, x_2, ..., x_k \in X$ such that $X = \bigcup\limits_{i=1}^{k} B^+(x_i, \delta_{x_i})$. By forward pointwise convergence of (f_n) to f, for each i there exist some $m_i \in \mathbb{N}$ such that for all $n \geq m_i$ we have $d_Y(f(x_i), f_n(x_i)) < \varepsilon/3$.

Now, take $n_0 = \max\{n_{x_1}, n_{x_2}, ..., n_{x_k}, m_1, ..., m_k\}$, and let $y \in X$ be arbitrary. So, $y \in B^+(x_i, \delta_{x_i})$ for some i. Then, $d_X(x_i, y) < \delta_{x_i} < \delta$. Hence, we obtain $d_Y(f(y), f(x_i)) < \varepsilon/3$ and $d_Y(f_n(x_i), f_n(y)) < \varepsilon/3$.

Therefore, for all $n \geq n_0$,

$$d_Y(f(y), f_n(y)) \leq d_Y(f(y), f(x_i)) + d_Y(f(x_i), f_n(x_i)) + d_Y(f_n(x_i), f_n(y)) < \varepsilon$$

Hence, $(f_n)_{n\in\mathbb{N}}$ is forward convergent uniformly to f. □

3. An Ascoli-type theorem in asymmetric metric spaces

The Arzelá-Ascoli theorem which characterizes compactness is a fundamental theorem of functional analysis. There can be found different statements for the Arzelá-Ascoli theorem in the literature (see in [10, 14, 21]). In [6], Collins and Zimmer prove the Arzelá-Ascoli theorem in the asymmetric case. Here, using exhaustiveness instead of equicontinuity in this theorem, we will give a type of the Ascoli theorem in the asymmetric context.

It is clear that, if we take $\mathcal{F}$ as a subset of $C(X, Y)$, then by Proposition 2.2. we can use exhaustiveness instead of equicontinuity in the Arzelá-Ascoli theorem in asymmetric case which was proved by Collins and Zimmer [6]. In that theorem, they gave relatively compactness of $\mathcal{F}$ under some conditions in asymmetric metric spaces. Now, with the following theorem we will give a characterization of compactness of $\mathcal{F}$ in asymmetric metric spaces.

First, we need the following lemma. In asymmetric metric spaces it can be given as in the metric case.

Lemma 3.1. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces, with forward convergence equivalent to backward convergence in Y. If Y is forward complete, then $C(X, Y)$ is forward complete in the uniform metric $\overline{\rho}$ corresponding to d_Y.*

Theorem 3.2. *Let X be forward compact, Y be forward complete asymmetric metric spaces and $\mathcal{F}$ be a subset of $C(X, Y)$. Suppose that forward convergence is equivalent to backward convergence in Y and every function sequence in $\mathcal{F}$ has a forward convergent subsequence in Y. Then $\mathcal{F}$ is forward compact if $\mathcal{F}$ is forward closed, forward totally bounded and forward exhaustive.*

Proof. We must show that, $\mathcal{F}$ has a function sequence containing a forward convergent subsequence. Let $(f_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{F}$ and $(x_n)_{n\in\mathbb{N}}$ be a forward dense subset of X. By assumption, the sequence $(f_n(x_1))_{n\in\mathbb{N}}$ has a subsequence $(f_{k_n^1}(x_1))_{n\in\mathbb{N}}$ which is forward convergent in Y. With the same argument, we can construct a subsequence $(f_{k_n^2}(x_2))_{n\in\mathbb{N}}$ of that subsequence which is forward convergent in Y.

Inductively, we obtain a sequence of sequences of naturals $...\subseteq (k_n^{j+1}) \subseteq (k_n^j) \subseteq ... \subseteq (k_n^1)$ such that for each $j \in \mathbb{N}$ the sequence $(f_{k_n^j}(x_j))_{n\in\mathbb{N}}$ is forward convergent in Y. It is clear that for each $j \in \mathbb{N}$ the diagonal sequence $(f_{k_n^n}(x_j))_{n\in\mathbb{N}}$ is forward convergent in Y. By forward convergence and implication of forward convergence to backward convergence in Y, this sequence is a forward Cauchy sequence. Since $(x_n)_{n\in\mathbb{N}}$ a forward dense subset of X, there is an $x_j \in X$ such that for every $x \in X$ and $j \in \{1, 2, ..., n\}$ we have $x_j \in B^+(x, \delta)$. By forward exhaustiveness of $\mathcal{F}$, for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $x \in X$ and for every $x_j \in B^+(x, \delta)$ and for every $k_n^n > n_0$ we have that $d_Y(f_{k_n^n}(x), f_{k_n^n}(x_j)) < \varepsilon$. Since $\mathcal{F} \subseteq C(X, Y)$ and forward convergence implies backward convergence in Y, we have that $d_Y(f_{k_n^n}(x_j), f_{k_n^n}(x)) < \varepsilon$. So, $\mathcal{F}$ is also backward exhaustive. Using these results, we obtain the sequence $(f_{k_n^n}(x))_{n\in\mathbb{N}}$ is a forward Cauchy sequence for each $x \in X$. Then by Lemma 3.1, $C(X, Y)$ is forward complete in the uniform metric $\overline{\rho}$ corresponding to d_Y and hence, the sequence $(f_{k_n^n}(x))_{n\in\mathbb{N}}$ is forward convergent.

Let $f_{k_n^n}(x) \xrightarrow{f} f(x)$. Then by forward exhaustiveness of $(f_{k_n^n}(x))_{n\in\mathbb{N}}$ and the forward compactness of X, using Theorem 2.9, we have that the sequence $(f_{k_n^n}(x))_{n\in\mathbb{N}}$ is forward convergent uniformly to f. Since $\mathcal{F}$ is forward closed, f belongs to $\mathcal{F}$ and so, $\mathcal{F}$ is forward sequentially compact. Finally, using forward sequentially compactness and forward total boundedness of $\mathcal{F}$, from Proposition 1.14, we find $\mathcal{F}$ is forward compact. □

4. Statistical version of exhaustiveness in asymmetric metric spaces

Caserta and Kočinac [3] studied the statistical version of exhaustiveness in metric spaces. In this section, we will investigate some properties of this notion in asymmetric metric spaces.

Definition 4.1. A sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ is said to be *forward statistically exhaustive* (shortly *forward st−exhaustive*) at a point $x \in X$ if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistically dense set $M \subset \mathbb{N}$ such that for each $y \in B^+(x,\delta)$ we have $d_Y(f_n(x), f_n(y)) < \varepsilon$ for each $n \in M$. The sequence $(f_n)_{n\in\mathbb{N}}$ is forward $st-$exhaustive if it is forward $st-$exhaustive at every $x \in X$.

Similarly, $(f_n)_{n\in\mathbb{N}} \subset Y^X$ is said to be *backward statistically exhaustive* (shortly *backward st−exhaustive*) at a point $x \in X$ if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistically dense set $M \subset \mathbb{N}$ such that for each $y \in B^+(x,\delta)$ we have $d_Y(f_n(y), f_n(x)) < \varepsilon$ for each $n \in M$. The sequence $(f_n)_{n\in\mathbb{N}}$ is backward $st-$exhaustive if it is backward $st-$exhaustive at every $x \in X$.

Remark 4.2. (1) Every forward equicontinuous function sequence $(f_n)_{n\in\mathbb{N}}$ is forward $st-$exhaustive, but the converse need not be true.

(2) Every forward exhaustive function sequence $(f_n)_{n\in\mathbb{N}}$ is forward st-exhaustive, but the converse need not be true.

Same remarks can be considered for backward ones.

With the following example we observe that there is a function sequence $(f_n)_{n\in\mathbb{N}}$ which is forward $st-$exhaustive. However, it is neither forward exhaustive nor forward equicontinuous.

Example 4.3. Let $X = \mathbb{R}$ and $Y = [-1, 1]$ be given with the following asymmetric metrics:

$$d_X(x,y) = \begin{cases} y-x, \text{ if } y \geq x \\ 1, \text{ if } y < x \end{cases} \quad \text{and} \quad d_Y(x,y) = \begin{cases} y-x, \text{ if } y \geq x \\ 2(x-y), \text{ if } y < x \end{cases}$$

Let us consider the function sequence $f_n : X \to Y, n \in \mathbb{N}$, defined in [3] by

$$f_n(x) = \begin{cases} -1, & \text{if } x \leq 0 \text{ and } n \text{ is prime} \\ \dfrac{1}{n}, & \text{if } x \leq 0 \text{ and } n \text{ is not prime} \\ 1, & \text{if } x > 0 \text{ and } n \text{ is prime} \\ \dfrac{1}{2n}, & \text{if } x > 0 \text{ and } n \text{ is not prime.} \end{cases}$$

The set of prime natural numbers P has asymptotic density $\partial(P) = 0$, take $\varepsilon > 0$ and $n_0 \in \mathbb{N}\backslash P$ such that $\dfrac{1}{n_0} < \varepsilon$. Then, for each $n \in (\mathbb{N}\backslash P) \cap \{n \in \mathbb{N} : n > n_0\}$ and $y \in B^+(0,\delta)$ we have $d_Y(f_n(0), f_n(y)) = 2\left(\dfrac{1}{n} - \dfrac{1}{2n}\right) < \varepsilon$. Hence, this function sequence is forward $st-$exhaustive at $x = 0$.

However, this function sequence is not forward exhaustive at $x = 0$. Indeed, for every $\delta > 0$ and for every $y \in B^+(0,\delta)$ we have $d_Y(f_n(0), f_n(y)) = f_n(y) - f_n(0) = 2$ for infinitely many n. Also, by the definition of forward equicontinuity and forward exhaustiveness, the sequence $(f_n)_{n\in\mathbb{N}}$ is not forward equicontinuous.

With the following example, it can be said that a function sequence is forward $st-$exhaustive if and only if backward $st-$exhaustive when forward $st-$convergence implies backward $st-$convergence in Y.

Example 4.4. Let us consider the previous example. The function sequence $(f_n)_{n\in\mathbb{N}}$ is also backward $st-$exhaustive at $x = 0$. Because, on the asymmetric metric Y, forward $st-$convergence implies backward $st-$convergence and $\partial(P) = 0$, taking again $\varepsilon > 0$ and $n_0 \in \mathbb{N}\backslash P$ such that $\dfrac{1}{n_0} < \varepsilon$, for each $n \in (\mathbb{N}\backslash P) \cap \{n \in \mathbb{N} : n > n_0\}$ and $y \in B^+(0,\delta)$, we have $d_Y(f_n(y), f_n(0)) = f_n(0) - f_n(y) = \dfrac{1}{n} - \dfrac{1}{2n} < \varepsilon$.

If we change the asymmetric metric on Y as in the following

$$d_Y(x,y) := \begin{cases} y-x, \text{ if } y \geq x \\ 1, \text{ if } y < x \end{cases}$$

the function sequence $(f_n)_{n\in\mathbb{N}}$ is not forward $st-$exhaustive at $x = 0$, however it is backward $st-$exhaustive at $x = 0$. Indeed, in this asymmetric metric, forward $st-$convergence does not equal to backward $st-$convergence and for every $\delta > 0$ and every $y \in B^+(0,\delta)$ we have $d_Y(f_n(y), f_n(0)) = f_n(0) - f_n(y) = \dfrac{1}{n} - \dfrac{1}{2n} < \varepsilon$, *i.e.*, $(f_n)_{n\in\mathbb{N}}$ is backward $st-$exhaustive at $x = 0$, but it is not forward $st-$exhaustive at $x \in X$, since for every $\delta > 0$ and every $y \in B^+(0,\delta)$ we have $d_Y(f_n(0), f_n(y)) = 1$.

Notice that we can give the following lemma in asymmetric case which is similar with the symmetric case given in [3].

Lemma 4.5. *A sequence $(f_n)_{n\in\mathbb{N}} \subseteq Y^X$ is forward (backward) $st-$exhaustive if and only if each of its $st-$dense subsequence is forward (backward) $st-$exhaustive.*

Definition 4.6. Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces and $(f_n)_{n\in\mathbb{N}}$ and f be functions defined from X to Y.

(1) We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *forward statistically pointwise convergent* (resp. *backward statistically pointwise convergent*) to f if for every $x \in X$ and every $\varepsilon > 0$ there exists a $st-$dense set $M \subset \mathbb{N}$ such that for all $n \in M$ we have $d_Y(f(x), f_n(x)) < \varepsilon$ (resp. $d(f_n(x), f(x)) < \varepsilon$).

(2) We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *forward statistically convergent uniformly* (resp. *backward statistically convergent uniformly*) with limit f if for every $\varepsilon > 0$, there exists a $st-$dense set $M \subset \mathbb{N}$ such that for all $n \in M$ and for every $x \in X$ we have $d_Y(f(x), f_n(x)) < \varepsilon$ (resp. $d(f_n(x), f(x)) < \varepsilon$).

As an important result of the definitions of statistical pointwise convergence and st-exhaustiveness of a function sequence in asymmetric metric spaces, the following proposition can be given.

Proposition 4.7. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces and $(f_n)_{n\in\mathbb{N}}$ and f be functions defined from X to Y. If $(f_n)_{n\in\mathbb{N}}$ is forward (backward) pointwise convergent to f and forward (backward) exhaustive, then $(f_n)_{n\in\mathbb{N}}$ is forward (backward) statistically pointwise convergent to f and forward (backward) $st-$exhaustive.*

Theorem 4.8. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. Assume that forward statistical convergence implies backward statistical convergence in Y. If $(f_n)_{n\in\mathbb{N}}$ is forward statistically pointwise convergent to f and forward $st-$exhaustive, then f is $ff-$continuous.*

Proof. Let $x \in X$ and $\varepsilon > 0$. Since $(f_n)_{n\in\mathbb{N}}$ is forward statistically pointwise convergent to f, there exists a $st-$dense set $K \subset \mathbb{N}$ such that for all $n \in K$ we have $d_Y(f(x), f_n(x)) < \varepsilon/3$. Take $y \in B^+(x,\delta)$. Then, there exists a $st-$dense set $L \subset \mathbb{N}$ such that for all $n \in L$ we have $d_Y(f_n(y), f(y)) < \varepsilon/3$, because of the convergence implication in Y. On the other hand, since the sequence $(f_n)_{n\in\mathbb{N}}$ is forward $st-$exhaustive at $x \in X$, there exists a $st-$dense set $M \subset \mathbb{N}$ such that for every $y \in B^+(x,\delta)$ and for every $n \in M$ we have $d_Y(f_n(x), f_n(y)) < \varepsilon/3$. Pick an arbitrary $j \in K \cap L \cap M$. Then, for $j \in K \cap L \cap M$ and $y \in B^+(x,\delta)$, we have

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f_j(x)) + d_Y(f_j(x), f_j(y)) + d_Y(f_j(y), f(y)) < \varepsilon.$$

It means that f is $ff-$continuous at x. □

Similarly, the following result can be given for $fb-$continuity of the forward statistical pointwise limit of the forward st-exhaustive function sequence.

Corollary 4.9. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. Assume that backward statistical convergence implies forward statistical convergence in Y. If $(f_n)_{n\in\mathbb{N}}$ is backward statistically pointwise convergent to f and backward $st-$exhaustive, then f is $fb-$continuous.*

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